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Convergence analysis of moving Godunov methods for dynamical boundary layers☆

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ABSTRACT

In this paper, a Godunov scheme on moving meshes is studied for kinds of time-dependent convection-dominated equations with dynamical boundary layers. The stability and a second-order spatial convergence are proved. A numerical example is provided to confirm the theoretical results.

1. Introduction

Consider a convection-dominated equation of the form

$$u_t - u_x - \epsilon u_{xx} = f(x,t), \quad \text{for } x \in (0,1),\ t \in [0,T], \tag{1}$$

$$u(x,0) = a(x), \quad \text{for } x \in [0,1], \tag{2}$$

$$u(0,t) = b_1(t), \qquad u(1,t) = b_2(t), \quad \text{for } t \in (0,T], \tag{3}$$

where $0 < \epsilon \ll 1$. Assume that $a \in C^2([0,1])$, $b_1, b_2 \in C^1([0,1])$, and the compatibility conditions

$$a(0) = b_1(0), \qquad a(1) = b_2(0) \tag{4}$$

hold true. With these assumptions, the solution of (1)–(3) exhibits boundary layers near the boundaries. Furthermore, the solution u can be decomposed into two parts, the smooth part w and the singular part v, i.e.,

$$u(x,t) = w(x,t) + v(x,t),$$

and there hold the following estimations (see [1,2]):

$$\left|\frac{\partial^{k+\ell} w}{\partial^k x \partial^\ell t}\right| \le C, \qquad \left|\frac{\partial^{k+\ell} v}{\partial^k x \partial^\ell t}\right| \approx C\left(1 + \epsilon^{-k} e^{-\gamma x/\epsilon}\right), \quad \text{for } k, \ell = 0, 1, 2, 3, \tag{5}$$

where $\gamma < 1$. Here and throughout the paper, C is independent of the parameter ϵ. The estimations

$$\begin{aligned} |u_{xx}| &\le C(1+u_x^2), \qquad & |u_{xxx}| &\le C(1+u_x^2)^{3/2}, \\ |u_{xt}| &\le C\sqrt{1+u_x^2}, \qquad & |u_{xtt}| &\le C\sqrt{1+u_x^2}, \end{aligned} \tag{6}$$

can be derived directly from (5).

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The solution u may also have one or more interior layers caused by the insufficient compatibility at the corner (0, 0) or the discontinuity of the initial function $a(x)$. A typical example is Eq. (1) for $x \in (-\infty, +\infty)$ with jumping initial condition $a(x) \equiv 0$ for $x < 0$ and $a(x) \equiv 1$ for $x > 0$, and then the solution has the form

$$u = \frac{1}{2} - \frac{1}{2}\operatorname{erfc}\left(\frac{x+t}{2\sqrt{\epsilon t}}\right),$$

where $\operatorname{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^{+\infty} e^{-\xi^2}\, d\xi$ is the error function. In this paper, we are not able to analyze the method for the interior layers, which is real challenge.

The Godunov scheme considered in this paper is based on the following moving mesh: define

$$\left\{0 \equiv x_0(t) < x_1(t) < \cdots < x_{N-1}(t) < x_N(t) \equiv 1\right\}$$

such that

$$\int_{x_j(t)}^{x_{j+1}(t)} m(\xi, t)\, d\xi = \frac{1}{N}\int_0^1 m(\xi, t)\, d\xi \quad (j = 0, 1, \dots, N-1), \tag{7}$$

where m is called the monitor function, which varies for different kinds of problems. In this paper, the monitor function is defined by

$$m(\xi, t) := \sqrt{1 + u_\xi^2}. \tag{8}$$

On the basis of the equidistribution principle (7), the relaxation methods have been used in practice (see e.g., Coyle, Flaherty and Ludwig [3]), and the moving mesh PDE approach introduced by Huang, Ren and Russell [4]. For a survey of the moving mesh methods, see, e.g., Huang and Russell [5], Zegeling [6], Tang [7] and Baines [8].

For ease of analysis and exposition, we consider problems (1)–(3) with $f \equiv 0$, $b_1 \equiv 0$, and $b_2 \equiv 0$.

We use the following brief notation in this paper:

$$x_j^n \equiv x_j(t_n); \qquad x_{j-1/2}^n \equiv \frac{x_{j-1}^n + x_j^n}{2}; \qquad \Delta x_j^n \equiv x_j(t_n) - x_{j-1}(t_n).$$

The moving mesh Godunov scheme is derived as follows. For convenience we multiply (1) (with $f \equiv 0$) by 2 and integrate it over the space-time domain $\Omega_j^n (j = 1, \dots, N-1;\ n = 0, 1, \dots, M-1)$; see Fig. 1. Then we have

$$\begin{aligned} 0 &= 2\int_{\Omega_j^n} (u_t - u_\xi - \epsilon u_{\xi\xi})\, d\xi d\tau = 2\left[\oint_{\partial\Omega_j^n} u\, d\xi + (u + \epsilon u_\xi)\, d\tau\right] \\ &= 2\left[\int_{x_{j-1/2}^{n+1}}^{x_{j+1/2}^{n+1}} u(\xi, t_{n+1})\, d\xi - \int_{x_{j-1/2}^{n}}^{x_{j+1/2}^{n}} u(\xi, t_n)\, d\xi\right] + 2\left[\int_{t_n}^{t_{n+1}} u(x_{j-1/2}(\tau), \tau) x'_{j-1/2}(\tau)\, d\tau\right. \\ &\quad \left.- \int_{t_n}^{t_{n+1}} u(x_{j+1/2}(\tau), \tau) x'_{j+1/2}(\tau)\, d\tau\right] + 2\left[\int_{t_n}^{t_{n+1}} u(x_{j-1/2}(\tau), \tau)\, d\tau - \int_{t_n}^{t_{n+1}} u(x_{j+1/2}(\tau), \tau)\, d\tau\right] \\ &\quad + 2\left[\int_{t_n}^{t_{n+1}} \epsilon u_x(x_{j-1/2}(\tau), \tau)\, d\tau - \int_{t_n}^{t_{n+1}} \epsilon u_x(x_{j+1/2}(\tau), \tau)\, d\tau\right]. \end{aligned} \tag{9}$$

Using a linear approximation to the mesh speed $x'(\tau)$ and backward finite difference for the derivatives u_ξ in the last two integrals, and then applying quadrature rules to the integrals gives a Godunov scheme. In this paper we analyze the following Godunov scheme which coincides with the Bonnerot–Jamet–Crank–Nicolson (BJCN) scheme described in [9,10]). The scheme is given by, for $j = 1, \dots, N-1$ and $n = 0, 1, \dots, M-1$,

$$\begin{aligned} & u_j^{n+1}[x_{j+1}^{n+1} - x_{j-1}^{n+1}] - u_j^n[x_{j+1}^n - x_{j-1}^n] - \frac{1}{2}\left[(u_{j+1}^{n+1} + u_{j+1}^n)(x_{j+1}^{n+1} - x_{j+1}^n) - (u_{j-1}^{n+1} + u_{j-1}^n)(x_{j-1}^{n+1} - x_{j-1}^n)\right] \\ & - \frac{\Delta t_n}{2}\left[(u_{j+1}^{n+1} - u_{j-1}^{n+1}) + (u_{j+1}^n - u_{j-1}^n)\right] - \epsilon\Delta t_n\left[\left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{x_{j+1}^{n+1} - x_j^{n+1}} - \frac{u_j^{n+1} - u_{j-1}^{n+1}}{x_j^{n+1} - x_{j-1}^{n+1}}\right)\right. \\ & \left. + \left(\frac{u_{j+1}^n - u_j^n}{x_{j+1}^n - x_j^n} - \frac{u_j^n - u_{j-1}^n}{x_j^n - x_{j-1}^n}\right)\right] = 0, \end{aligned} \tag{10}$$

where we define $\Delta t_n = t_{n+1} - t_n$ and make use of the boundary and initial conditions

$$u_0^n = u_N^n = 0, \qquad u_j^0 = a(x_j^0).$$

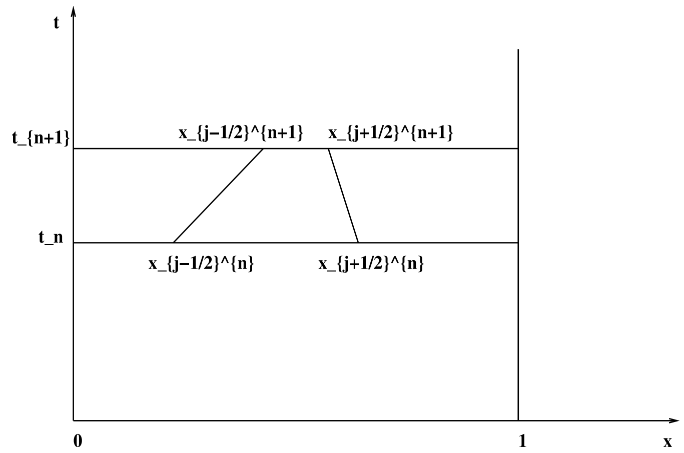

Fig. 1. Demonstration of Ω_j^n.

In this paper, we prove stability and a second-order spatial and first-order temporal convergence of the Godunov scheme for problems (1)–(3) with $f \equiv 0$, $b_1 \equiv 0$, and $b_2 \equiv 0$.

It is instructive to give a brief review of the convergence theory of moving mesh methods for related elliptic and time-dependent problems.

For elliptic convection-dominated equations, a lot of significant work has been done recently on convergence using the equidistribution principle for selecting the meshes. The papers of Qiu, Sloan and Tang [11] and Beckett and Mackenzie [12] study the uniform convergence of the upwind finite difference methods. Chen and Xu [13] investigate the uniform convergence of the finite element methods for one-dimensional elliptic convection-dominated equations. Huang [14] analyzes the uniform convergence of the finite element methods using the approximate equidistributing meshes which are constructed via the monitor function upper bounding the polynomial interpolation error. The works by Kopteva [15] and Kopteva and Stynes [16] prove the uniform convergence in maximum norms of the upwind finite difference methods on the approximate equidistributing meshes that are determined by the computed solutions.

However for time-dependent problems, especially for convection-dominated problems, little advance has been made so far. In an earlier paper [10], the authors give a review and numerical study of three moving mesh algorithms — the BJCN scheme (it can be regarded as a special case of the Godunov methods), the IEL scheme (the implicit Euler Lagrangian scheme), and RFDM (the rezoning finite difference method). The BJCN scheme was first introduced in [9]. Jamet [17] gives a convergence proof for the heat equation with moving boundaries, where the mesh movement is driven by the moving boundaries while keeping the spatial mesh uniform at each time level. Their proof is highly reliant on the uniformity of the spatial mesh in each time level, and the technique cannot be used for a variable mesh in each time level. In this paper we give a proof of the Godunov scheme for equations with dominant convection terms based on an equidistributing moving mesh. Our proof gives a new framework that can be essentially used for more general Godunov schemes. Mackenzie and Mekwi [18] prove an asymptotic second-order convergence for a conservative IEL scheme.

2. Analysis of the moving mesh strategy

In this section, we derive several properties that the equidistributing moving mesh (7) can satisfy.

Theorem 2.1. *For the moving strategy (7), we have the following estimations:*

(i) $x_{j+1}(t) - x_j(t) \le C/N$; (ii) $|u_x(x_j(t), t)x_j'(t)| \le C$;
(iii) $|x_j'(t)| \le C$; (iv) $|x_j''(t)| \le C$.

Here and throughout this paper, C denotes a generic positive constant that is independent of the parameter ϵ.

Proof. Since

$$x_{j+1}(t) - x_j(t) = \int_{x_j(t)}^{x_{j+1}(t)} \mathrm{d}\xi \le \int_{x_j(t)}^{x_{j+1}(t)} m(\xi, t)\, \mathrm{d}\xi \le \frac{1}{N}\int_0^1 m(\xi, t)\, \mathrm{d}\xi,$$

and

$$\int_0^1 m(\xi, t)\, \mathrm{d}\xi \le C, \quad \text{(from (5))},$$

the proof of (i) is complete.

To prove (ii), we take the derivative in t for the form equivalent to (7):

$$\int_0^{x_j(t)} m(\xi,t)\,\mathrm{d}\xi = \frac{j}{N}\int_0^1 m(\xi,t)\,\mathrm{d}\xi,$$

and then this gives

$$m(x_j(t),t)x_j'(t) = \frac{j}{N}\int_0^1 m_t(\xi,t)\,\mathrm{d}\xi - \int_0^{x_j(t)} m_t(\xi,t)\,\mathrm{d}\xi. \tag{11}$$

Consequently,

$$|u_x(x_j(t),t)x_j'(t)| \le |m(x_j(t),t)x_j'(t)| \le 2\int_0^1 |m_t(\xi,t)|\,\mathrm{d}\xi, \tag{12}$$

and

$$|x_j'(t)| \le |m(x_j(t),t)x_j'(t)| \le 2\int_0^1 |m_t(\xi,t)|\,\mathrm{d}\xi.$$

Taking into account (5), we are led to

$$\int_0^1 |m_t(\xi,t)|\,\mathrm{d}\xi = \int_0^1 \frac{|u_\xi u_{\xi t}|}{\sqrt{1+u_\xi^2}}\,\mathrm{d}\xi \le \int_0^1 |u_{\xi t}|\,\mathrm{d}\xi \le C. \tag{13}$$

Thus, the proof of (ii) and (iii) is done.

Taking the derivative of (11) gives

$$m(x_j(t),t)x_j''(t) = \frac{j}{N}\int_0^1 m_{tt}(\xi,t)\,\mathrm{d}\xi - \int_0^{x_j(t)} m_{tt}(\xi,t)\,\mathrm{d}\xi - 2m_t(x_j(t),t)x_j'(t) - m_x(x_j(t),t)(x_j'(t))^2. \tag{14}$$

To derive the upper bound for $x_j''(t)$, we estimate for

$$\frac{m_{tt}(\xi,t)}{m(x_j(t),t)},\qquad \frac{m_t(x_j(t),t)x_j'(t)}{m(x_j(t),t)},\quad \text{and}\quad \frac{m_x(x_j(t),t)x_j'(t)}{m(x_j(t),t)}$$

as follows.

$$\begin{aligned}\left|\frac{m_{tt}(\xi,t)}{m(x_j(t),t)}\right| &\le |m_{tt}(\xi,t)| = \left|\frac{(u_{\xi t})^2 + u_\xi(1+u_\xi^2)u_{\xi tt}}{(1+u_\xi^2)^{3/2}}\right| \\ &= \left|\frac{u_{\xi t}(u_{\xi\xi}+\epsilon u_{\xi\xi\xi})}{(1+u_\xi^2)^{3/2}} + \frac{u_\xi u_{\xi tt}}{(1+u_\xi^2)^{1/2}}\right| \\ &\le C(|u_{\xi t}| + |u_{\xi tt}|),\end{aligned} \tag{15}$$

where in the last inequality we applied (6). Applying (ii), (6), and (5) gives

$$\begin{aligned}\left|\frac{m_t(x_j(t),t)x_j'(t)}{m(x_j(t),t)}\right| &= \left|\frac{u_{xt}}{1+u_x^2}\right| |u_x(x_j(t),t)x_j'(t)| \\ &\le C\left|\frac{u_{xt}}{1+u_x^2}\right| = C\left|\frac{u_{xx}+\epsilon u_{xxx}}{1+u_x^2}\right| \\ &\le C + C\epsilon\sqrt{1+u_x^2} \le C.\end{aligned} \tag{16}$$

Similarly we derive

$$\left|\frac{m_x(x_j(t),t)x_j'(t)}{m(x_j(t),t)}\right| = \left|\frac{u_{xx}}{1+u_x^2}\right| |u_x(x_j(t),t)x_j'(t)| \le C. \tag{17}$$

Combining (15)–(17) into (14), we obtain the desired assertion (iv). $\square$

In the analysis of this paper, we will also use the following conditions which are commonly used by other authors in their works, e.g., [17,18].

For small time step Δt_n, the monitor function does not vary extremely; then we have the mesh condition:

$$1 - C\Delta t_n \le \frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{x_{j+1}^{n} - x_{j-1}^{n}} \le 1 + C\Delta t_n, \tag{18}$$

for all available indices j and n.

We assume that the time step is sufficiently small to satisfy the CFL condition

$$\epsilon\Delta t_n \le C(\Delta x_j^n)^2, \tag{19}$$

for all available indices j and n.

Moreover we assume that the mesh is locally smooth such that

$$|(x_{j+1}(t) - x_j(t)) - (x_j(t) - x_{j-1}(t))| \le C \max\left\{(x_{j+1}(t) - x_j(t))^2, (x_j(t) - x_{j-1}(t))^2\right\}. \tag{20}$$

To have more transparency, we note that for equidistributing mesh it holds that

$$\int_{x_j(t)}^{x_{j+1}(t)} m(x,t)\,\mathrm{d}x - \int_{x_{j-1}}^{x_j} m(x,t)\,\mathrm{d}x = 0.$$

Use

$$m(x,t) = m(x_j(t),t) + m_x(x_j(t),t)(x - x_j(t)) + \text{H.O.T}$$

to give

$$(x_{j+1}(t) - x_j(t)) - (x_j(t) - x_{j-1}(t)) = \frac{m_x(x_j(t),t)}{m(x_j(t),t)}\left[\left(\frac{x_{j+1}(t) - x_j(t)}{2}\right)^2 + \left(\frac{x_j(t) - x_{j-1}(t)}{2}\right)^2\right] + \text{H.O.T.}$$

Then

$$|(x_{j+1}(t) - x_j(t)) - (x_j(t) - x_{j-1}(t))| \le C\left|\frac{m_x(x_j(t),t)}{m(x_j(t),t)}\right|\left[\left(\frac{x_{j+1}(t) - x_j(t)}{2}\right)^2 + \left(\frac{x_j(t) - x_{j-1}(t)}{2}\right)^2\right].$$

So it is required that the local term $\left|\frac{m_x(x_j(t),t)}{m(x_j(t),t)}\right|$ is bounded to make (20) be satisfied. In practice, a locally smoothing technique (see e.g., Huang and Russell [19]) can supply this need.

3. Stability

Write (10) in an equivalent matrix form:

$$\mathbf{A}^{n+1}\mathrm{U}^{n+1} = \mathbf{B}^n\mathrm{U}^n \quad (n = 0, 1, \ldots, M-1), \tag{21}$$

where

$$\mathrm{U}^n := (\mathrm{u}_1^n, \ldots, \mathrm{u}_{N-1}^n)^{\mathrm{T}},$$

and

$$\mathbf{A}^{n+1} = \left(a_{i,j}^{n+1}\right)_{i,j=1,\ldots,N-1}, \tag{22}$$

$$a_{j,j}^{n+1} = (x_{j+1}^{n+1} - x_{j-1}^{n+1}) + \frac{\epsilon\Delta t_n}{\Delta x_j^{n+1}} + \frac{\epsilon\Delta t_n}{\Delta x_{j+1}^{n+1}}, \quad (j = 1, \ldots, N-1);$$

$$a_{j,j-1}^{n+1} = \frac{x_{j-1}^{n+1} - x_{j-1}^{n}}{2} + \frac{\Delta t_n}{2} - \frac{\epsilon\Delta t_n}{\Delta x_j^{n+1}} \quad (j = 1, \ldots, N-1);$$

$$a_{j,j+1}^{n+1} = -\frac{x_{j+1}^{n+1} - x_{j+1}^{n}}{2} - \frac{\Delta t_n}{2} - \frac{\epsilon\Delta t_n}{\Delta x_{j+1}^{n+1}} \quad (j = 1, \ldots, N-1);$$

$$a_{i,j}^{n+1} = 0, \quad \text{otherwise},$$

and

$$\mathbf{B}^n = \left(b^n_{i,j}\right)_{i,j=1,\dots,N-1}, \tag{23}$$

$$b^n_{j,j} = (x^n_{j+1} - x^n_{j-1}) - \frac{\epsilon\Delta t_n}{\Delta x^n_j} - \frac{\epsilon\Delta t_n}{\Delta x^n_{j+1}}, \quad (j = 1,\dots,N-1);$$

$$b^n_{j,j-1} = -\frac{x^{n+1}_{j-1} - x^n_{j-1}}{2} - \frac{\Delta t_n}{2} + \frac{\epsilon\Delta t_n}{\Delta x^n_j} \quad (j = 1,\dots,N-1);$$

$$b^n_{j,j+1} = \frac{x^{n+1}_{j+1} - x^n_{j+1}}{2} + \frac{\Delta t_n}{2} + \frac{\epsilon\Delta t_n}{\Delta x^n_{j+1}} \quad (j = 1,\dots,N-1);$$

$$b^n_{i,j} = 0, \quad \text{otherwise.}$$

Define a mesh-dependent L_2-norm:

$$\|v\|_n := \left(\sum_j v^2(x^n_j)\frac{x^n_{j+1} - x^n_{j-1}}{2}\right)^{1/2}.$$

Let $\mathrm{V} = (v(x_1(t_n)),\dots,v(x_{N-1}(t_n)))^{\mathrm{T}}$. Then the following norms are equivalent:

$$\|v\|_n \equiv \left\|\mathrm{diag}\left(\sqrt{\frac{x^n_{j+1} - x^n_{j-1}}{2}}\right)\mathrm{V}\right\|_2.$$

For vectors, $\|\cdot\|_2$ denotes the Euclidean norm, while for matrices, $\|\cdot\|_2$ denotes the matrix spectral norm.

In the following theorem, we derive the stability for the system (21).

Theorem 3.1. *The stability for system (21) is given with*

$$\left\|\mathrm{diag}\left(\sqrt{\frac{x^{n+1}_{j+1} - x^{n+1}_{j-1}}{2}}\right)\mathrm{U}^{n+1}\right\|_2 \le C\left\|\mathrm{diag}\left(\sqrt{\frac{x^0_{j+1} - x^0_{j-1}}{2}}\right)\mathrm{U}^0\right\|_2 \tag{24}$$

under the mesh conditions (19), (18) and assumption $\Delta x^{n+1}_j \le \epsilon/C$.

Proof. We take inner product of (21) by U^{n+1}:

$$\left\langle \mathbf{A}^{n+1}\mathrm{U}^{n+1}, \mathrm{U}^{n+1}\right\rangle = \left\langle \mathbf{B}^n\mathrm{U}^n, \mathrm{U}^{n+1}\right\rangle. \tag{25}$$

We estimate (25) term by term. We first estimate the LHS. Since $a^{n+1}_{j,i} = 0$ for $|j-i| > 1$, we obtain that

$$\left\langle \mathbf{A}^{n+1}\mathrm{U}^{n+1}, \mathrm{U}^{n+1}\right\rangle = \sum_j a^{n+1}_{j,j}\left(\mathrm{u}^{n+1}_j\right)^2 + \sum_j a^{n+1}_{j,j-1}\mathrm{u}^{n+1}_j\mathrm{u}^{n+1}_{j-1} + \sum_j a^{n+1}_{j,j+1}\mathrm{u}^{n+1}_j\mathrm{u}^{n+1}_{j+1}$$
$$\text{for all } \mathrm{U}^{n+1} := (\mathrm{u}^{n+1}_1,\dots,\mathrm{u}^{n+1}_{N-1})^{\mathrm{T}} \ne 0. \tag{26}$$

Shifting the index for the last summation in the RHS of equality (26), we obtain that

$$\begin{aligned}
&\left|\sum_j a^{n+1}_{j,j-1}\mathrm{u}^{n+1}_j\mathrm{u}^{n+1}_{j-1} + \sum_j a^{n+1}_{j,j+1}\mathrm{u}^{n+1}_j\mathrm{u}^{n+1}_{j+1}\right| = \left|\sum_j (a^{n+1}_{j,j-1} + a^{n+1}_{j-1,j})\mathrm{u}^{n+1}_j\mathrm{u}^{n+1}_{j-1}\right| \\
&\quad\le \sum_j \frac{|a^{n+1}_{j,j-1} + a^{n+1}_{j-1,j}|}{2}\left(\mathrm{u}^{n+1}_j\right)^2 + \sum_j \frac{|a^{n+1}_{j,j-1} + a^{n+1}_{j-1,j}|}{2}\left(\mathrm{u}^{n+1}_{j-1}\right)^2 \\
&\quad= \sum_j \frac{|a^{n+1}_{j,j-1} + a^{n+1}_{j-1,j}| + |a^{n+1}_{j+1,j} + a^{n+1}_{j,j+1}|}{2}\left(\mathrm{u}^{n+1}_j\right)^2 \\
&\quad\le \sum_j \left[\frac{(-x^{n+1}_j + x^{n+1}_{j-1}) + (x^n_j - x^n_{j-1})}{4} + \frac{(-x^{n+1}_{j+1} + x^{n+1}_j) + (x^n_{j+1} - x^n_j)}{4} + \frac{\epsilon\Delta t_n}{\Delta x^{n+1}_j} + \frac{\epsilon\Delta t_n}{\Delta x^{n+1}_{j+1}}\right]\left(\mathrm{u}^{n+1}_j\right)^2 \\
&\quad= \sum_i \left[-\frac{x^{n+1}_{j+1} - x^{n+1}_{j-1}}{4} + \frac{x^n_{j+1} - x^n_{j-1}}{4} + \frac{\epsilon\Delta t_n}{\Delta x^{n+1}_j} + \frac{\epsilon\Delta t_n}{\Delta x^{n+1}_{j+1}}\right]\left(\mathrm{u}^{n+1}_j\right)^2,
\end{aligned} \tag{27}$$

where in the second inequality, we have used (18) and made an assumption that $\Delta x_j^{n+1} \le \epsilon/C$. Therefore we obtain that

$$\begin{aligned}\left\langle \mathbf{A}^{n+1}\mathrm{U}^{n+1}, \mathrm{U}^{n+1}\right\rangle &\ge \sum_i \left[(x_{j+1}^{n+1} - x_{j-1}^{n+1}) - \frac{-(x_{j+1}^{n+1} - x_{j-1}^{n+1}) + (x_{j+1}^{n} - x_{j-1}^{n})}{4} \right] \left(\mathrm{u}_j^{n+1}\right)^2 \\ &= \sum_i \left[\frac{5}{4}(x_{j+1}^{n+1} - x_{j-1}^{n+1}) - \frac{1}{4}(x_{j+1}^{n} - x_{j-1}^{n}) \right] \left(\mathrm{u}_j^{n+1}\right)^2 . \end{aligned} \tag{28}$$

Now we estimate the second term of (25):

$$\begin{aligned}\left|\left\langle \mathbf{B}^{n}\mathrm{U}^{n}, \mathrm{U}^{n+1}\right\rangle\right| &= \left| \sum_j (b_{j,j-1}^n \mathrm{u}_{j-1}^n + b_{j,j}^n \mathrm{u}_j^n + b_{j,j+1}^n \mathrm{u}_{j+1}^n)\mathrm{u}_j^{n+1} \right| \\ &\le \sum_j \left[\frac{|b_{j,j-1}^n|}{2}(\mathrm{u}_{j-1}^n)^2 + \frac{|b_{j,j}^n|}{2}(\mathrm{u}_j^n)^2 + \frac{|b_{j,j+1}^n|}{2}(\mathrm{u}_{j+1}^n)^2 \right] + \sum_j \frac{|b_{j,j-1}^n| + |b_{j,j}^n| + |b_{j,j+1}^n|}{2}(\mathrm{u}_j^{n+1})^2 \\ &= \sum_j \left[\frac{|b_{j+1,j}^n|}{2} + \frac{|b_{j,j}^n|}{2} + \frac{|b_{j-1,j}^n|}{2} \right](\mathrm{u}_j^n)^2 + \sum_j \frac{|b_{j,j-1}^n| + |b_{j,j}^n| + |b_{j,j+1}^n|}{2}(\mathrm{u}_j^{n+1})^2. \end{aligned} \tag{29}$$

Provided the CFL condition $\epsilon\Delta t_n \le C(\Delta x_j^n)^2$ for all available index j, for sufficiently small time step Δt_n, it holds that $b_{j,j}^n \ge 0$. Moreover since $|x_j^{n+1} - x_j^n| \le C\Delta t_n$ (see (iii) in Theorem 2.1), we know that $b_{j,j-1}^n,\ b_{j,j+1}^n \ge 0$ for all indexes j. Therefore,

$$\sum_j \left[\frac{|b_{j+1,j}^n|}{2} + \frac{|b_{j,j}^n|}{2} + \frac{|b_{j-1,j}^n|}{2} \right](\mathrm{u}_j^n)^2 = \sum_j \frac{x_{j+1}^n - x_{j-1}^n}{2}(\mathrm{u}_j^n)^2, \tag{30}$$

and

$$\begin{aligned}&\sum_j \frac{|b_{j,j-1}^n| + |b_{j,j}^n| + |b_{j,j+1}^n|}{2}(\mathrm{u}_j^{n+1})^2 \\ &\quad= \sum_j \left[\frac{1}{4}(x_{j+1}^{n+1} - x_{j-1}^{n+1}) + \frac{1}{4}(x_{j+1}^{n} - x_{j-1}^{n}) \right](\mathrm{u}_j^{n+1})^2. \end{aligned} \tag{31}$$

We finally get

$$\left|\left\langle \mathbf{B}^{n}\mathrm{U}^{n}, \mathrm{U}^{n+1}\right\rangle\right| \le \sum_j \frac{x_{j+1}^n - x_{j-1}^n}{2}(\mathrm{u}_j^n)^2 + \sum_j \left[\frac{1}{4}(x_{j+1}^{n+1} - x_{j-1}^{n+1}) + \frac{1}{4}(x_{j+1}^{n} - x_{j-1}^{n}) \right](\mathrm{u}_j^{n+1})^2. \tag{32}$$

Incorporating estimations (28) and (32) into (25) gives that

$$\sum_j \left[(x_{j+1}^{n+1} - x_{j-1}^{n+1}) - \frac{1}{2}(x_{j+1}^{n} - x_{j-1}^{n}) \right](\mathrm{u}_j^{n+1})^2 \tag{33}$$

$$\le \sum_j \frac{x_{j+1}^n - x_{j-1}^n}{2}(\mathrm{u}_j^n)^2 + \frac{\Delta t_n}{2} \left\| \mathrm{diag}\left(\sqrt{\frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}} \right) \mathrm{U}^{n+1} \right\|_2^2 . \tag{34}$$

Using the mesh condition (18), we obtain that

$$(x_{j+1}^{n+1} - x_{j-1}^{n+1}) - \frac{1}{2}(x_{j+1}^{n} - x_{j-1}^{n}) \ge \frac{1 - C\Delta t_n}{1 + C\Delta t_n} \frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}. \tag{35}$$

Hence,

$$\left\| \mathrm{diag}\left(\sqrt{\frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}} \right) \mathrm{U}^{n+1} \right\|_2^2 \le \frac{1 + C\Delta t_n}{1 - C\Delta t_n} \left[\left\| \mathrm{diag}\left(\sqrt{\frac{x_{j+1}^{n} - x_{j-1}^{n}}{2}} \right) \mathrm{U}^{n} \right\|_2^2 . \right.$$

Finally using the iteration and taking into account

$$\frac{1 + C\Delta t_n}{1 - C\Delta t_n} \cdots \frac{1 + C\Delta t_0}{1 - C\Delta t_0} \le C, \tag{36}$$

we get the desired result (24). $\square$

4. Convergence

We first estimate the truncation error.

Define the truncation error as

$$\mathcal{T}_j^n = \text{RHS of (9)} - \text{LHS of (10) with } u_j^n \text{ replaced by } u(x_j^n, t_n), \tag{37}$$

for $j = 1, \ldots, N-1$ and $n = 0, 1, \ldots, M-1$. To derive the expression for $\mathcal{T}_j^n$, we use Taylor theory to estimate the error of numerical integrations of the integrals in (9). We first derive that

$$\begin{aligned} &\int_{x_{j-1/2}^{n+1}}^{x_{j+1/2}^{n+1}} u(\xi, t_{n+1})\, \mathrm{d}\xi - \int_{x_{j-1/2}^{n}}^{x_{j+1/2}^{n}} u(\xi, t_n)\, \mathrm{d}\xi \\ &\quad = u(x_j^{n+1}, t_{n+1}) \frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2} - u(x_j^n, t_n) \frac{x_{j+1}^n - x_{j-1}^n}{2} + \frac{\mathrm{T}_1}{2}, \end{aligned} \tag{38}$$

where

$$\mathrm{T}_1 = 2\int_{x_{j-1/2}^{n+1}}^{x_{j+1/2}^{n+1}} u_\xi(x_j^{n+1}, t_{n+1})(\xi - x_j^{n+1})\, \mathrm{d}\xi - 2\int_{x_{j-1/2}^{n}}^{x_{j+1/2}^{n}} u_\xi(x_j^n, t_n)(\xi - x_j^n)\, \mathrm{d}\xi + \text{H.O.T.} \tag{39}$$

Since

$$u(x_{j-1}(\tau), \tau) = u(x_{j-1/2}(\tau), \tau) + u_\xi(x_{j-1/2}(\tau), \tau)(x_{j-1}(\tau) - x_{j-1/2}(\tau)) + \text{H.O.T.},$$

and

$$u(x_j(\tau), \tau) = u(x_{j-1/2}(\tau), \tau) + u_\xi(x_{j-1/2}(\tau), \tau)(x_j(\tau) - x_{j-1/2}(\tau)) + \text{H.O.T.},$$

we obtain

$$\begin{aligned} &\int_{t_n}^{t_{n+1}} u(x_{j-1/2}(\tau), \tau) x'_{j-1/2}(\tau)\, \mathrm{d}\tau \\ &\quad = \int_{t_n}^{t_{n+1}} \frac{u(x_{j-1}(\tau), \tau) x'_{j-1}(\tau) + u(x_j(\tau), \tau) x'_j(\tau)}{2}\, \mathrm{d}\tau \\ &\qquad - \int_{t_n}^{t_{n+1}} u_\xi(x_{j-1/2}(\tau), \tau) \frac{(x_j(\tau) - x_{j-1}(\tau))}{2} \frac{(x'_j(\tau) - x'_{j-1}(\tau))}{2}\, \mathrm{d}\tau + \Delta t_n\, (\text{H.O.T.})\,. \end{aligned} \tag{40}$$

Similarly we can obtain

$$\begin{aligned} &\int_{t_n}^{t_{n+1}} u(x_{j+1/2}(\tau), \tau) x'_{j+1/2}(\tau)\, \mathrm{d}\tau \\ &\quad = \int_{t_n}^{t_{n+1}} \frac{u(x_j(\tau), \tau) x'_j(\tau) + u(x_{j+1}(\tau), \tau) x'_{j+1}(\tau)}{2}\, \mathrm{d}\tau \\ &\qquad - \int_{t_n}^{t_{n+1}} u_\xi(x_{j+1/2}(\tau), \tau) \frac{(x_{j+1}(\tau) - x_j(\tau))}{2} \frac{(x'_{j+1}(\tau) - x'_j(\tau))}{2}\, \mathrm{d}\tau + \Delta t_n\, (\text{H.O.T.})\,. \end{aligned} \tag{41}$$

Therefore we arrive at

$$\begin{aligned} &\int_{t_n}^{t_{n+1}} u(x_{j-1/2}(\tau), \tau) x'_{j-1/2}(\tau)\, \mathrm{d}\tau - \int_{t_n}^{t_{n+1}} u(x_{j+1/2}(\tau), \tau) x'_{j+1/2}(\tau)\, \mathrm{d}\tau \\ &\quad = \int_{t_n}^{t_{n+1}} \frac{u(x_{j-1}(\tau), \tau) x'_{j-1}(\tau) - u(x_{j+1}(\tau), \tau) x'_{j+1}(\tau)}{2}\, \mathrm{d}\tau \\ &\qquad - \int_{t_n}^{t_{n+1}} u_\xi(x_{j-1/2}(\tau), \tau) \frac{(x_j(\tau) - x_{j-1}(\tau))}{2} \frac{(x'_j(\tau) - x'_{j-1}(\tau))}{2}\, \mathrm{d}\tau \\ &\qquad + \int_{t_n}^{t_{n+1}} u_\xi(x_{j+1/2}(\tau), \tau) \frac{(x_{j+1}(\tau) - x_j(\tau))}{2} \frac{(x'_{j+1}(\tau) - x'_j(\tau))}{2}\, \mathrm{d}\tau + \Delta t_n\, (\text{H.O.T.})\,. \end{aligned} \tag{42}$$

The first term of the RHS is given by

$$\int_{t_n}^{t_{n+1}} \frac{u(x_{j-1}(\tau),\tau)x'_{j-1}(\tau) - u(x_{j+1}(\tau),\tau)x'_{j+1}(\tau)}{2}\,\mathrm{d}\tau \tag{43}$$
$$= -\frac{1}{4}\left[(u(x_{j+1}^{n+1},t_{n+1}) + u(x_{j+1}^{n},t_n))(x_{j+1}^{n+1} - x_{j+1}^{n}) - (u(x_{j-1}^{n+1},t_{n+1}) + u(x_{j-1}^{n},t_n))(x_{j-1}^{n+1} - x_{j-1}^{n})\right]$$
$$+ \frac{1}{2}\int_{t_n}^{t_{n+1}}\int_{t_n}^{\tau}\left(\frac{u(x_{j-1}(\eta),\eta)x'_{j-1}(\eta) - u(x_{j+1}(\eta),\eta)x'_{j+1}(\eta)}{2}\right)'\,\mathrm{d}\eta\mathrm{d}\tau$$
$$+ \frac{1}{2}\int_{t_n}^{t_{n+1}}\int_{t_{n+1}}^{\tau}\left(\frac{u(x_{j-1}(\eta),\eta)x'_{j-1}(\eta) - u(x_{j+1}(\eta),\eta)x'_{j+1}(\eta)}{2}\right)'\,\mathrm{d}\eta\mathrm{d}\tau,$$

where we made a use of

$$x'_j(\tau) = \frac{x_j^{n+1} - x_j^n}{\Delta t_n} \quad \text{for } \tau \in (t_n, t_{n+1}).$$

Therefore we arrive at

$$\int_{t_n}^{t_{n+1}} u(x_{j-1/2}(\tau),\tau)x'_{j-1/2}(\tau)\,\mathrm{d}\tau - \int_{t_n}^{t_{n+1}} u(x_{j+1/2}(\tau),\tau)x'_{j+1/2}(\tau)\,\mathrm{d}\tau \tag{44}$$
$$= -\frac{1}{4}\left[(u(x_{j+1}^{n+1},t_{n+1}) + u(x_{j+1}^{n},t_n))(x_{j+1}^{n+1} - x_{j+1}^{n}) - (u(x_{j-1}^{n+1},t_{n+1}) + u(x_{j-1}^{n},t_n))(x_{j-1}^{n+1} - x_{j-1}^{n})\right] + \frac{\mathrm{T}_2}{2},$$

where

$$\mathrm{T}_2 = \int_{t_n}^{t_{n+1}}\int_{t_n}^{\tau}\left(\frac{u(x_{j-1}(\eta),\eta)x'_{j-1}(\eta) - u(x_{j+1}(\eta),\eta)x'_{j+1}(\eta)}{2}\right)'\,\mathrm{d}\eta\mathrm{d}\tau \tag{45}$$
$$+ \int_{t_n}^{t_{n+1}}\int_{t_{n+1}}^{\tau}\left(\frac{u(x_{j-1}(\eta),\eta)x'_{j-1}(\eta) - u(x_{j+1}(\eta),\eta)x'_{j+1}(\eta)}{2}\right)'\,\mathrm{d}\eta\mathrm{d}\tau$$
$$-2\int_{t_n}^{t_{n+1}} u_\xi(x_{j-1/2}(\tau),\tau)\frac{(x_j(\tau) - x_{j-1}(\tau))}{2}\frac{(x'_j(\tau) - x'_{j-1}(\tau))}{2}\,\mathrm{d}\tau$$
$$+2\int_{t_n}^{t_{n+1}} u_\xi(x_{j+1/2}(\tau),\tau)\frac{(x_{j+1}(\tau) - x_j(\tau))}{2}\frac{(x'_{j+1}(\tau) - x'_j(\tau))}{2}\,\mathrm{d}\tau + \Delta t_n\,(\text{H.O.T.})\,.$$

We also derive that

$$\int_{t_n}^{t_{n+1}} u(x_{j-1/2}(\tau),\tau)\,\mathrm{d}\tau - \int_{t_n}^{t_{n+1}} u(x_{j+1/2}(\tau),\tau)\,\mathrm{d}\tau \tag{46}$$
$$= \int_{t_n}^{t_{n+1}} \frac{u(x_{j-1}(\tau),\tau) - u(x_{j+1}(\tau),\tau)}{2}\,\mathrm{d}\tau$$
$$+ \int_{t_n}^{t_{n+1}} u_{\xi\xi}(x_{j-1/2}(\tau),\tau)\frac{(x_j(\tau) - x_{j-1}(\tau))^2}{8}\,\mathrm{d}\tau - \int_{t_n}^{t_{n+1}} u_{\xi\xi}(x_{j+1/2}(\tau),\tau)\frac{(x_{j+1}(\tau) - x_j(\tau))^2}{8}\,\mathrm{d}\tau + \text{H.O.T.}$$

We then achieve that

$$\int_{t_n}^{t_{n+1}} u(x_{j-1/2}(\tau),\tau)\,\mathrm{d}\tau - \int_{t_n}^{t_{n+1}} u(x_{j+1/2}(\tau),\tau)\,\mathrm{d}\tau \tag{47}$$
$$= -\frac{\Delta t_n}{4}\left[(u_{j+1}^{n+1} - u_{j-1}^{n+1}) + (u_{j+1}^{n} - u_{j-1}^{n})\right] + \frac{\mathrm{T}_3}{2},$$

where

$$\mathrm{T}_3 = \int_{t_n}^{t_{n+1}}\int_{t_n}^{\tau}\left(\frac{u(x_{j-1}(\eta),\eta) - u(x_{j+1}(\eta),\eta)}{2}\right)'\,\mathrm{d}\eta\mathrm{d}\tau \tag{48}$$
$$+ \int_{t_n}^{t_{n+1}}\int_{t_{n+1}}^{\tau}\left(\frac{u(x_{j-1}(\eta),\eta) - u(x_{j+1}(\eta),\eta)}{2}\right)'\,\mathrm{d}\eta\mathrm{d}\tau + \int_{t_n}^{t_{n+1}} u_{\xi\xi}(x_{j-1/2}(\tau),\tau)\frac{(x_j(\tau) - x_{j-1}(\tau))^2}{8}\,\mathrm{d}\tau$$
$$- \int_{t_n}^{t_{n+1}} u_{\xi\xi}(x_{j+1/2}(\tau),\tau)\frac{(x_{j+1}(\tau) - x_j(\tau))^2}{8}\,\mathrm{d}\tau + \Delta t_n\,(\text{H.O.T.})\,.$$

Now we estimate the last two integrals in (9). To this end, we derive that

$$u(x_j(\tau),\tau) = u(x_{j-1/2}(\tau),\tau) + u_\xi(x_{j-1/2}(\tau),\tau)(x_j(\tau)-x_{j-1/2}(\tau)) + \frac{1}{2}u_{\xi\xi}(x_{j-1/2}(\tau),\tau)(x_j(\tau)-x_{j-1/2}(\tau))^2 + \frac{1}{6}u_{\xi\xi\xi}(x_{j-1/2}(\tau),\tau)(x_j(\tau)-x_{j-1/2}(\tau))^3 + \text{H.O.T.},$$

and

$$u(x_{j-1}(\tau),\tau) = u(x_{j-1/2}(\tau),\tau) + u_\xi(x_{j-1/2}(\tau),\tau)(x_{j-1}(\tau)-x_{j-1/2}(\tau)) + \frac{1}{2}u_{\xi\xi}(x_{j-1/2}(\tau),\tau)(x_{j-1}(\tau)-x_{j-1/2}(\tau))^2 + \frac{1}{6}u_{\xi\xi\xi}(x_{j-1/2}(\tau),\tau)(x_{j-1}(\tau)-x_{j-1/2}(\tau))^3 + \text{H.O.T.}$$

The subtraction of the above two identities gives that

$$u_\xi(x_{j-1/2}(\tau),\tau) = \frac{u(x_j(\tau),\tau)-u(x_{j-1}(\tau),\tau)}{x_j(\tau)-x_{j-1}(\tau)} - \frac{1}{6}u_{\xi\xi\xi}(x_{j-1/2}(\tau),\tau)\left(\frac{x_j(\tau)-x_{j-1}(\tau)}{2}\right)^2 + \text{H.O.T.}$$

By the same argument, we can obtain that

$$u_\xi(x_{j+1/2}(\tau),\tau) = \frac{u(x_{j+1}(\tau),\tau)-u(x_j(\tau),\tau)}{x_{j+1}(\tau)-x_j(\tau)} - \frac{1}{6}u_{\xi\xi\xi}(x_{j+1/2}(\tau),\tau)\left(\frac{x_{j+1}(\tau)-x_j(\tau)}{2}\right)^2 + \text{H.O.T.}$$

Therefore we finally arrive at

$$\begin{aligned}
&\int_{t_n}^{t_{n+1}} \epsilon u_x(x_{j-1/2}(\tau),\tau)\,\mathrm{d}\tau - \int_{t_n}^{t_{n+1}} \epsilon u_x(x_{j+1/2}(\tau),\tau)\,\mathrm{d}\tau \qquad (49)\\
&\quad = \epsilon\int_{t_n}^{t_{n+1}}\left[\frac{u(x_j(\tau),\tau)-u(x_{j-1}(\tau),\tau)}{x_j(\tau)-x_{j-1}(\tau)} - \frac{u(x_{j+1}(\tau),\tau)-u(x_j(\tau),\tau)}{x_{j+1}(\tau)-x_j(\tau)}\right]\mathrm{d}\tau\\
&\qquad - \frac{\epsilon}{6}\int_{t_n}^{t_{n+1}} u_{\xi\xi\xi}(x_{j-1/2}(\tau),\tau)\left(\frac{x_j(\tau)-x_{j-1}(\tau)}{2}\right)^2\mathrm{d}\tau\\
&\qquad + \frac{\epsilon}{6}\int_{t_n}^{t_{n+1}} u_{\xi\xi\xi}(x_{j+1/2}(\tau),\tau)\left(\frac{x_{j+1}(\tau)-x_j(\tau)}{2}\right)^2\mathrm{d}\tau + \Delta t_n\,(\text{H.O.T.})\,, \qquad (50)
\end{aligned}$$

or

$$\begin{aligned}
&\int_{t_n}^{t_{n+1}} \epsilon u_x(x_{j-1/2}(\tau),\tau)\,\mathrm{d}\tau - \int_{t_n}^{t_{n+1}} \epsilon u_x(x_{j+1/2}(\tau),\tau)\,\mathrm{d}\tau \qquad (51)\\
&\quad = -\frac{\epsilon\Delta t_n}{2}\left[\left(\frac{u(x_{j+1}^{n+1},t_{n+1})-u(x_j^{n+1},t_{n+1})}{x_{j+1}^{n+1}-x_j^{n+1}} - \frac{u(x_j^{n+1},t_{n+1})-u(x_{j-1}^{n+1},t_{n+1})}{x_j^{n+1}-x_{j-1}^{n+1}}\right)\right.\\
&\qquad \left.+\left(\frac{u(x_{j+1}^n,t_n)-u(x_j^n,t_n)}{x_{j+1}^n-x_j^n} - \frac{u(x_j^n,t_n)-u(x_{j-1}^n,t_n)}{x_j^n-x_{j-1}^n}\right)\right] + \frac{\mathrm{T}_4}{2},
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{T}_4 &= \epsilon\int_{t_n}^{t_{n+1}}\int_{t_n}^{\tau}\left[\frac{u(x_j(\eta),\eta)-u(x_{j-1}(\eta),\eta)}{x_j(\eta)-x_{j-1}(\eta)} - \frac{u(x_{j+1}(\eta),\eta)-u(x_j(\eta),\eta)}{x_{j+1}(\eta)-x_j(\eta)}\right]'\mathrm{d}\eta\mathrm{d}\tau\\
&\quad + \epsilon\int_{t_n}^{t_{n+1}}\int_{t_{n+1}}^{\tau}\left[\frac{u(x_j(\eta),\eta)-u(x_{j-1}(\eta),\eta)}{x_j(\eta)-x_{j-1}(\eta)} - \frac{u(x_{j+1}(\eta),\eta)-u(x_j(\eta),\eta)}{x_{j+1}(\eta)-x_j(\eta)}\right]'\mathrm{d}\eta\mathrm{d}\tau\\
&\quad - \frac{\epsilon}{3}\int_{t_n}^{t_{n+1}} u_{\xi\xi\xi}(x_{j-1/2}(\tau),\tau)\left(\frac{x_j(\tau)-x_{j-1}(\tau)}{2}\right)^2\mathrm{d}\tau\\
&\quad + \frac{\epsilon}{3}\int_{t_n}^{t_{n+1}} u_{\xi\xi\xi}(x_{j+1/2}(\tau),\tau)\left(\frac{x_{j+1}(\tau)-x_j(\tau)}{2}\right)^2\mathrm{d}\tau + \Delta t_n\,(\text{H.O.T.})\,, \qquad (52)
\end{aligned}$$

Taking into account of (38), (44), (47) and (51), we obtain the expression for $\mathcal{T}_j^n$:

$$\mathcal{T}_j^n := \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3 + \mathrm{T}_4, \qquad (53)$$

where $\mathrm{T}_1,\ \mathrm{T}_2,\ \mathrm{T}_3,\ \mathrm{T}_4$ are given by (39), (45), (48) and (52), respectively.

Now we derive the upper bounds on $T_1,\ T_2,\ T_3,\ T_4$ as follows.
We estimate

$$
\begin{aligned}
|T_1| = {} & \left| u_\xi(x_j^{n+1}, t_{n+1}) \left[\left(\frac{x_{j+1}^{n+1} - x_j^{n+1}}{2} \right)^2 - \left(\frac{x_j^{n+1} - x_{j-1}^{n+1}}{2} \right)^2 \right] \right. \\
& \left. - \ u_\xi(x_j^n, t_n) \left[\left(\frac{x_{j+1}^n - x_j^n}{2} \right)^2 - \left(\frac{x_j^n - x_{j-1}^n}{2} \right)^2 \right] \right| + \text{H.O.T.} \\
= {} & \left| \int_{t_n}^{t_{n+1}} \left\{ u_\xi(x_j(\tau), \tau) \left[\left(\frac{x_{j+1}(\tau) - x_j(\tau)}{2} \right)^2 - \left(\frac{x_j(\tau) - x_{j-1}(\tau)}{2} \right)^2 \right] \right\}' \mathrm{d}\tau \right| + \text{H.O.T.}
\end{aligned} \tag{54}
$$

Then making use of (20) and (18), we can obtain that

$$
|T_1| \le C \max \left\{ (x_{j+1}^{n+1} - x_{j-1}^{n+1}), (x_{j+1}^n - x_{j-1}^n) \right\} \frac{\Delta t_n}{N^2}. \tag{55}
$$

To estimate the first two terms of T_2, using (11), Theorem 2.1, (6) and (5), we derive that

$$
\begin{aligned}
& \left| u_x(x_{j-1}(\tau), \tau) x_{j-1}'(\tau) - u_x(x_{j+1}(\tau), \tau) x_{j+1}'(\tau) \right| \\
& = \left| \frac{u_x(x_{j-1}(\tau), \tau)}{m(x_{j-1}(\tau), \tau)} \left(\frac{j-1}{N} \int_0^1 m_\tau(\xi, \tau)\, \mathrm{d}\xi - \int_0^{x_{j-1}(\tau)} m_\tau(\xi, \tau)\, \mathrm{d}\xi \right) \right. \\
& \quad \left. - \ \frac{u_x(x_{j+1}(\tau), \tau)}{m(x_{j+1}(\tau), \tau)} \left(\frac{j+1}{N} \int_0^1 m_\tau(\xi, \tau)\, \mathrm{d}\xi - \int_0^{x_{j+1}(\tau)} m_\tau(\xi, \tau)\, \mathrm{d}\xi \right) \right| \\
& \le \left| \frac{u_x(x_{j+1}(\tau), \tau)}{m(x_{j+1}(\tau), \tau)} \left(\frac{1}{N} \int_0^1 m_\tau(\xi, \tau)\, \mathrm{d}\xi - \int_{x_{j-1}(\tau)}^{x_{j+1}(\tau)} m_\tau(\xi, \tau)\, \mathrm{d}\xi \right) \right| \\
& \quad + \left| \frac{u_x(x_{j+1}(\tau), \tau)}{m(x_{j+1}(\tau), \tau)} - \frac{u_x(x_j(\tau), \tau)}{m(x_j(\tau), \tau)} \right| \left| \frac{j-1}{N} \int_0^1 m_\tau(\xi, \tau)\, \mathrm{d}\xi - \int_0^{x_{j-1}(\tau)} m_\tau(\xi, \tau)\, \mathrm{d}\xi \right| \\
& \le \frac{C}{N} + C \int_{x_{j-1}(\tau)}^{x_{j+1}(\tau)} \left| \left(\frac{u_\xi(\xi, \tau)}{m(\xi, \tau)} \right)_\xi \right| \mathrm{d}\xi \\
& = \frac{C}{N} + C \int_{x_{j-1}(\tau)}^{x_{j+1}(\tau)} \left| \frac{u_{\xi\xi}(\xi, \tau)}{m(\xi, \tau)} - \frac{(u_\xi(\xi, \tau))^2 u_{\xi\xi}(\xi, \tau)}{(m(\xi, \tau))^3} \right| \mathrm{d}\xi \\
& \le \frac{C}{N} + C \int_{x_{j-1}(\tau)}^{x_{j+1}(\tau)} \left| \frac{u_{\xi\xi}(\xi, \tau)}{m(\xi, \tau)} \right| \mathrm{d}\xi \\
& \le \frac{C}{N} + C \int_{x_{j-1}(\tau)}^{x_{j+1}(\tau)} m(\xi, \tau)\, \mathrm{d}\xi \le \frac{C}{N},
\end{aligned} \tag{56}
$$

and

$$
\left| u_\tau(x_{j+1}(\tau), \tau) - u_\tau(x_{j-1}(\tau), \tau) \right| \le \int_{x_{j-1}(\tau)}^{x_{j+1}(\tau)} u_{\xi\tau}(\xi, \tau)\, \mathrm{d}\xi \le \frac{C}{N}. \tag{57}
$$

Therefore, it follows from (56) and (57) that

$$
\left| \int_{t_n}^{t_{n+1}} \int_{t_n}^{\eta} \big(u(x_{j-1}(\tau), \tau) \big)' \,\mathrm{d}\tau \mathrm{d}\eta - \int_{t_n}^{t_{n+1}} \int_{t_n}^{\eta} \big(u(x_{j+1}(\tau), \tau) \big)' \,\mathrm{d}\tau \mathrm{d}\eta \right| \le \frac{C(\Delta t_n)^2}{N}. \tag{58}
$$

Like in the derivation of (58), we have

$$
\left| \int_{t_n}^{t_{n+1}} \int_{\eta}^{t_{n+1}} \big(u(x_{j-1}(\tau), \tau) \big)' \,\mathrm{d}\tau \mathrm{d}\eta - \int_{t_n}^{t_{n+1}} \int_{\eta}^{t_{n+1}} \big(u(x_{j+1}(\tau), \tau) \big)' \,\mathrm{d}\tau \mathrm{d}\eta \right| \le \frac{C(\Delta t_n)^2}{N}. \tag{59}
$$

Hence,

$$
|\text{First Two Terms of } T_2| \le C \frac{(\Delta t_n)^2}{N}. \tag{60}
$$

Furthermore, using the mesh conditions (18) and (20) gives that

$$|\text{Last Three Terms of } \mathrm{T}_2| \le C \max\left\{(x_{j+1}^{n+1} - x_{j-1}^{n+1}), (x_{j+1}^{n} - x_{j-1}^{n})\right\} \frac{\Delta t_n}{N^2}. \tag{61}$$

Using (58) and (59) for the first two terms of T_3, and (20) for the last two, gives that

$$|\mathrm{T}_3| \le C\frac{(\Delta t_n)^2}{N} + C\max\left\{(x_{j+1}^{n+1} - x_{j-1}^{n+1}), (x_{j+1}^{n} - x_{j-1}^{n})\right\} \frac{\Delta t_n}{N^2}. \tag{62}$$

Using Theorem 2.1, (5) and the analogous derivation of (58), we obtain the estimation for the first two terms of T_4. Applying (20) directly gives the estimation for the last three terms. That is

$$|\mathrm{T}_4| \le C\frac{(\Delta t_n)^2}{N} + C\max\left\{(x_{j+1}^{n+1} - x_{j-1}^{n+1}), (x_{j+1}^{n} - x_{j-1}^{n})\right\} \frac{\Delta t_n}{N^2}. \tag{63}$$

As a conclusion from the estimations (55) and (60)–(63), we obtain that

$$\left|\mathcal{T}_j^n\right| \le C\frac{(\Delta t_n)^2}{N} + C\max\left\{(x_{j+1}^{n+1} - x_{j-1}^{n+1}), (x_{j+1}^{n} - x_{j-1}^{n})\right\} \frac{\Delta t_n}{N^2}. \tag{64}$$

Now we are in a position to present the convergence result.

Theorem 4.1. *Under the same assumptions as for Theorem 3.1, the error of the Godunov scheme (10) for the unsteady convection-dominated equation (1)–(3) with $f \equiv 0$, $b_1 \equiv 0$, and $b_2 \equiv 0$ is bounded by*

$$\|u(x, t_n) - u^n(x)\|_n \le C\left[N^{-2} + \max_{(n)}\{\Delta t_n\}\right],$$

where C is independent of the small diffusion parameter ϵ.

Proof. Let $\mathrm{E}^n := (\mathrm{e}_1^n, \ldots, \mathrm{e}_{N-1}^n)^{\mathrm{T}} = (u(x_1(t_n), t_n) - \mathrm{u}_1^n, \ldots, u(x_{N-1}(t_n), t_n) - \mathrm{u}_{N-1}^n)^{\mathrm{T}}$ and $\mathrm{T}^n := (\mathcal{T}_1^n, \ldots, \mathcal{T}_{N-1}^n)^{\mathrm{T}}$. Then we write

$$\mathbf{A}^{n+1}\mathrm{E}^{n+1} = \mathbf{B}^n\mathrm{E}^n + \mathrm{T}^n \quad (n = 0, 1, \ldots, M-1), \tag{65}$$

with $\mathrm{E}^0 \equiv 0$. We take the inner product of (65) with E^{n+1}:

$$\left\langle \mathbf{A}^{n+1}\mathrm{E}^{n+1}, \mathrm{E}^{n+1}\right\rangle = \left\langle \mathbf{B}^n\mathrm{E}^n, \mathrm{E}^{n+1}\right\rangle + \left\langle \mathrm{T}^n, \mathrm{E}^{n+1}\right\rangle. \tag{66}$$

The estimations of the LHS and the first term of RHS in (66) are given by (28) and (32), respectively (with the symbol U replaced by E). Now we estimate the third term of (66):

$$\begin{aligned}\left\langle \mathrm{T}^n, \mathrm{E}^{n+1}\right\rangle &\le \left\|\mathrm{diag}^{-1}\left(\sqrt{\frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}}\right)\mathrm{T}^n\right\|_2 \left\|\mathrm{diag}\left(\sqrt{\frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}}\right)\mathrm{E}^{n+1}\right\|_2 \\ &\le \frac{1}{2\Delta t_n}\left\|\mathrm{diag}^{-1}\left(\sqrt{\frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}}\right)\mathrm{T}^n\right\|_2^2 + \frac{\Delta t_n}{2}\left\|\mathrm{diag}\left(\sqrt{\frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}}\right)\mathrm{E}^{n+1}\right\|_2^2. \end{aligned} \tag{67}$$

Incorporating estimations (28), (32) (with the symbol U replaced by E) and (67) in (66) gives that

$$\sum_j \left[(x_{j+1}^{n+1} - x_{j-1}^{n+1}) - \frac{1}{2}(x_{j+1}^{n} - x_{j-1}^{n})\right](\mathrm{e}_j^{n+1})^2 \tag{68}$$

$$\le \sum_j \frac{x_{j+1}^{n} - x_{j-1}^{n}}{2}(\mathrm{e}_j^n)^2 + \frac{1}{2\Delta t_n}\left\|\mathrm{diag}^{-1}\left(\sqrt{\frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}}\right)\mathrm{T}^n\right\|_2^2 + \frac{\Delta t_n}{2}\left\|\mathrm{diag}\left(\sqrt{\frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}}\right)\mathrm{E}^{n+1}\right\|_2^2. \tag{69}$$

Using (35) we obtain that

$$\begin{aligned}\left\|\mathrm{diag}\left(\sqrt{\frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}}\right)\mathrm{E}^{n+1}\right\|_2^2 \le \frac{1 + C\Delta t_n}{1 - C\Delta t_n}\Bigg[&\left\|\mathrm{diag}\left(\sqrt{\frac{x_{j+1}^{n} - x_{j-1}^{n}}{2}}\right)\mathrm{E}^n\right\|_2^2 \\ &+ \frac{1}{2\Delta t_n}\left\|\mathrm{diag}^{-1}\left(\sqrt{\frac{x_{j+1}^{n+1} - x_{j-1}^{n+1}}{2}}\right)\mathrm{T}^n\right\|_2^2\Bigg].\end{aligned}$$

Finally using the iteration and taking into account (36), we get

$$\left\|\operatorname{diag}\left(\sqrt{\frac{x_{j+1}^{n+1}-x_{j-1}^{n+1}}{2}}\right)\mathrm{E}^{n+1}\right\|_2 \le C\left\|\operatorname{diag}\left(\sqrt{\frac{x_{j+1}^{0}-x_{j-1}^{0}}{2}}\right)\mathrm{E}^{0}\right\|_2 + C\left[\sum_{k=1}^{n}(\Delta t_k)^{-1}\left\|\operatorname{diag}^{-1}\left(\sqrt{\frac{x_{j+1}^{k+1}-x_{j-1}^{k+1}}{2}}\right)\mathrm{T}^{k}\right\|_2^2\right]^{1/2}. \tag{70}$$

Applying (70) and (64) completes the proof. □

5. Numerical example

We consider a simple example to test the convergence rate.

$$u_t - u_x - \epsilon u_{xx} = f(x,t), \quad x \in (0,1),\ t \in [0, \pi/2], \tag{71}$$

$$u(0,t) = u(1,t) = 0, \tag{72}$$

$$u(x,0) = 0, \tag{73}$$

where

$$f = \left[x\cos(xt) - t\cos(xt) + \epsilon t^2 \sin(xt)\right]\tanh\left(\frac{1-x}{\epsilon}\right) + \left[\frac{\sin(xt)}{\epsilon} + 2t\cos(xt) + \frac{2}{\epsilon}\sin(xt)\tanh\left(\frac{1-x}{\epsilon}\right)\right]\operatorname{sech}^2\left(\frac{1-x}{\epsilon}\right).$$

The exact solution is

$$u = \sin(xt)\tanh\left(\frac{1-x}{\epsilon}\right).$$

Consider scheme (10) with the RHS replaced by

$$\mathrm{RHS} = \frac{1}{2}\Delta t_n\left[(x_{j+1}^n - x_{j-1}^n)f(x_j^n, t_n) + (x_{j+1}^{n+1} - x_{j-1}^{n+1})f(x_j^{n+1}, t_{n+1})\right], \tag{74}$$

for $j = 1, \dots, N-1$ and $n = 0, 1, \dots, M-1$,

In the test we use the arc-length monitor function

$$\widetilde{m}(x,t) = \sqrt{1+u_x^2}$$

with a locally smoothing procedure

$$m(x,t) = \frac{1}{3}\Big[\widetilde{m}(x-2\epsilon,t) + \widetilde{m}(x,t) + \widetilde{m}(x+2\epsilon,t)\Big].$$

Let N denote the number of the spatial mesh intervals and M the number of the temporal mesh intervals. Define the rate of convergence: Rate $= \log_2\left(\mathrm{Error}(N_{j+1})/\mathrm{Error}(N_j)\right) / \log_2\left(N_j/N_{j+1}\right)$. The numerics in Tables 1 and 2 confirm the second-order convergence results.

6. Conclusion

In this paper we have proved the second-order convergence for a Godunov scheme on an equidistributing moving mesh for a convection-dominated problem. Our proof is the first step towards understanding the moving mesh methods assuming that the exact solutions are used in generating the moving mesh. However in practice the numerical solutions are used in generating the mesh. The proof, which will use the a posteriori analysis, is the second step of building up the theory of moving mesh methods for solving time-dependent PDEs.

Although we used a linear approximation to the mesh speed $x'(t)$ in the scheme, it is not essentially difficult to obtain a proof for high-order approximation of the mesh speed, which is related to construction of a high-order scheme. The framework of our proof can be applied to more general Godunov schemes and BJCN-type schemes. The derivation of Godunov schemes readily allows the systematic construction of high-order schemes. The investigation of high-order schemes will be left for future work.

In the condition (19), the time step Δt_n is required to be very small in the layer region. This difficulty can be overcome by a technique of locally varying the time step expounded in [20], but the analysis remains open. In addition, a practical local technique for smoothing the monitor functions needs to be designed to guarantee that the local term $\left|\frac{m_x(x_j(t),t)}{m(x_j(t),t)}\right|$ is bounded and thus guarantee the mesh condition (20).

Table 1
$\epsilon = 0.001$.

N	M	$\max_n \|e^n\|_n$	Rate
50	100	1.9335×10^{-2}	2.7
100	200	2.9399×10^{-3}	2.8
200	400	4.2036×10^{-4}	2.5
400	800	7.0140×10^{-5}	N/A

Table 2
$\varepsilon = 0.0001$.

N	M	$\max_n \|e^n\|_n$	Rate
50	100	2.4866×10^{-1}	3.1
100	200	2.7387×10^{-2}	2.6
200	400	4.5114×10^{-3}	2.4
400	800	8.0043×10^{-4}	N/A

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